Attractor Formation Requires Entropy Export: A Thermodynamic Necessity Argument for the AHT Equations

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Abstract

We show that any system governed by the AHT field-Laplacian equations that forms stable attractors must necessarily export entropy to its environment. The argument proceeds in five steps: we establish the thermodynamic openness of the system, derive a Lyapunov stability condition, connect Lyapunov descent to net dissipation, prove that the closed-system case prohibits non-trivial attractors, and derive a quantitative lower bound on the entropy export rate. The exact dissipation balance $J_{\text{export}} = T^{-1}(\gamma\|\psi_*\|^2 + \kappa\|\delta L_*\|_F^2)$ relates the entropy export rate to the attractor geometry. We briefly discuss the biological implication for neural energy budgets.

Status: Mathematical memo. Steps 1–4 constitute rigorous arguments under stated assumptions; Step 5 is a formal bound; Step 6 is interpretive. Proofs are marked as such; heuristic arguments are labelled “Argument.”

1 Setup and Notation

Consider a weighted graph $\mathcal{G} = (V, E, W)$ with $n = |V|$ nodes and graph Laplacian $L_0 = D - W$. The AHT dynamics are:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\mathrm{i}(L_0 + \delta L)\,\psi - \gamma\psi + S(t), \tag{1}$$

$$\frac{\mathrm{d}(\delta L)}{\mathrm{d}t} = -\eta\,F(t)\;\mathrm{Re}[\psi\psi^\dagger] - \kappa\,\delta L, \tag{2}$$

where $\psi \in \mathbb{C}^n$, $\delta L \in \mathbb{R}^{n\times n}_{\text{sym}}$, $\gamma > 0$ (field damping), $\kappa > 0$ (forgetting rate), $\eta > 0$ (learning rate), $F(t) \in [0, 1]$ (neuromodulatory gate), and $S(t) \in \mathbb{C}^n$ (external input).

We suppress the bistable nonlinearity $f_{\text{nl}}(\psi)$ throughout. All results hold when f_{nl} is dissipative (i.e. $\mathrm{Re}\langle f_{\text{nl}}(\psi), \psi\rangle \le 0$); including it strengthens the inequalities.

Assumption 1 (Finite energy)**.** *$\|\psi(t)\|^2 < \infty$ and $\|\delta L(t)\|_F^2 < \infty$ for all $t \ge 0$.*

Assumption 2 (Bounded input)**.** *There exists $P_S < \infty$ such that $\mathrm{Re}\langle S(t), \psi(t)\rangle \le P_S$ for all t.*

2 Step 1: Energy Balance of the Open System

Proposition 1 (Field energy balance)**.** *Define the field energy* $E_\psi(t) = \frac{1}{2}\|\psi\|^2$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\psi = \underbrace{\mathrm{Re}\langle S(t), \psi\rangle}_{P_{\mathrm{in}}} - \underbrace{\gamma\|\psi\|^2}_{P_\gamma} - \underbrace{\mathrm{Im}\langle \delta L\, \psi, \psi\rangle}_{P_{\delta L}}. \tag{3}$$

Proof. Compute $\frac{\mathrm{d}}{\mathrm{d}t}E_\psi = \mathrm{Re}\langle\dot{\psi}, \psi\rangle$. Substituting (1):

$$\mathrm{Re}\langle\dot{\psi}, \psi\rangle = \mathrm{Re}\langle -\mathrm{i}(L_0 + \delta L)\psi - \gamma\psi + S, \psi\rangle.$$

Since L_0 is real symmetric, $\mathrm{Re}\langle -\mathrm{i}L_0\psi, \psi\rangle = -\mathrm{Im}\langle L_0\psi, \psi\rangle = -\mathrm{Im}(\psi^\dagger L_0 \psi) = 0$ because $\psi^\dagger L_0\psi \in \mathbb{R}$. The δL term contributes $-\mathrm{Im}\langle \delta L\, \psi, \psi\rangle$, which is nonzero only when δL has time-dependent asymmetric coupling with the complex phase of ψ. The damping term gives $-\gamma\|\psi\|^2$, and the input contributes $\mathrm{Re}\langle S, \psi\rangle$. ☐

Remark 1. *When* δL *is real symmetric (as constructed by (2)), the term* $P_{\delta L}$ *vanishes identically:* $\psi^\dagger(\delta L)\psi \in \mathbb{R}$ *implies* $\mathrm{Im}\langle \delta L\, \psi, \psi\rangle = 0$*. The field energy balance then simplifies to the clean form*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\psi = P_{\mathrm{in}} - P_\gamma, \qquad P_\gamma = \gamma\|\psi\|^2. \tag{4}$$

The system absorbs energy at rate P_{in} *and dissipates at rate* P_γ*. No energy is stored in the wave-Laplacian coupling.*

Proposition 2 (Laplacian energy balance)**.** *Define the Laplacian energy* $E_L(t) = \frac{1}{2}\|\delta L\|_F^2$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_L = \underbrace{-\eta\, F(t)\ \mathrm{tr}[\delta L \cdot \mathrm{Re}[\psi\psi^\dagger]]}_{P_{\mathrm{Hebb}}} - \underbrace{\kappa\|\delta L\|_F^2}_{P_\kappa}. \tag{5}$$

Proof. $\frac{\mathrm{d}}{\mathrm{d}t}E_L = \mathrm{tr}[\delta L^\top \cdot \dot{\delta L}]$. Substituting (2) and using symmetry of δL yields (5) directly. ☐

Assumption 3 (Physical interpretation of E_L)**.** *The* κ*-decay term* $-\kappa\, \delta L$ *represents a physical dissipation process (e.g. synaptic turnover, metabolic cost of maintaining potentiated synapses). The associated power* $P_\kappa = \kappa\|\delta L\|_F^2$ *is dissipated as heat into the thermal environment and contributes to entropy export. The quantity* $E_L = \frac{1}{2}\|\delta L\|_F^2$ *tracks the free energy stored in the structural perturbation; its units are fixed by choosing* δL *in units of (energy)*$^{1/2}$ *per node.*

Remark 2 (Total energy budget)**.** *The total system energy* $E = E_\psi + E_L$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}E = P_{\mathrm{in}} + P_{\mathrm{Hebb}} - P_\gamma - P_\kappa. \tag{6}$$

Energy enters via the input $S(t)$ *and the Hebbian pump* P_{Hebb} *(which can be positive or negative); energy exits via field damping* P_γ *and Laplacian decay* P_κ*. This establishes the system as thermodynamically open: it requires sustained energy flow to maintain a non-trivial state.*

3 Step 2: Attractor Condition via Lyapunov Function

Assumption 4 (Attractor stationarity)**.** *A stable attractor* $(\psi_*, \delta L_*)$ *exists such that at the attractor,* $\dot{\psi} \approx 0$ *and* $\dot{\delta L} \approx 0$ *under constant input* S_**.*

At the attractor, the stationarity conditions read:

$$0 = -\mathrm{i}(L_0 + \delta L_*)\psi_* - \gamma\psi_* + S_*, \tag{7}$$

$$0 = -\eta\, F_*\ \mathrm{Re}[\psi_*\psi_*^\dagger] - \kappa\, \delta L_*. \tag{8}$$

Define the Lyapunov candidate:

$$\mathcal{F}(\psi, \delta L) \;=\; \frac{1}{2}\|\dot{\psi}\|^2 + \frac{\kappa}{2\eta}\|\delta L\|_F^2. \tag{9}$$

Proposition 3 (Lyapunov descent)**.** *Under Assumptions 1–4, if* $\gamma > 0$ *and* $\kappa > 0$*, then in a neighbourhood of the attractor:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \;\leq\; 0, \tag{10}$$

with equality only at the fixed point $(\psi_*, \delta L_*)$*.*

Argument (not a complete proof; see Bean (2026) for the full computation). The first term $\frac{1}{2}\|\dot{\psi}\|^2$ decreases when the effective dissipation γ drains 'prediction error' (the residual $\dot{\psi}$) faster than new perturbations enter. The second term decreases at rate $-\kappa^2\|\delta L\|_F^2/\eta$ from the κ-decay. The cross-terms from the Hebbian coupling $\dot{\delta L} \to \dot{\psi}$ are bounded by the damping when γ is sufficiently large relative to $\eta F(t)\|\psi\|^2$. The precise condition is:

$$\gamma \;>\; \frac{\eta\, F_{\max}\, \|\psi_*\|^2}{2\kappa}\, \|\nabla_\psi [(L_0 + \delta L_*)\psi_*]\|, \tag{11}$$

which is the requirement that damping dominates the learning-induced feedback. This is the regime in which the brain normally operates: $\gamma/\kappa \gg \eta/\gamma$. □

The key consequence of $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$ is:

Corollary 1 (Net dissipation)**.** *At a stable attractor with* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$*:*

$$P_\gamma + P_\kappa \;=\; P_{\mathrm{in}} + P_{\mathrm{Hebb}} \quad \textit{(at the fixed point)}, \tag{12}$$

and $P_\gamma + P_\kappa > P_{\mathrm{in}} + P_{\mathrm{Hebb}}$ *for all states in the basin of attraction except the fixed point.*

Discussion. The equality at the fixed point is immediate from $\frac{\mathrm{d}}{\mathrm{d}t}E = 0$ (stationarity) and (6). The strict inequality throughout the basin is the content of the Lyapunov condition: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$ implies that the composite state $\mathcal{F}$ decreases toward the fixed point, which requires that the net energy balance is negative away from it. The precise connection between $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$ and the energy inequality for arbitrary perturbations involves computing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}$ in terms of P_γ, P_κ, and cross-terms; this is carried out in Bean (2026) for the linearised system. □

4 Step 3: Dissipation as Entropy Export

We now connect the energy-level results to thermodynamic entropy, following Prigogine's formulation for open systems (Prigogine, 1967).

Assumption 5 (Local equilibrium)**.** *The environment (thermal bath) into which the system dissipates energy is at temperature* $T > 0$*. The dissipated energy is thermalised irreversibly.*

Under Assumption 5, the entropy production splits as:

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{\mathrm{sys}} \;=\; \sigma_{\mathrm{int}} \;-\; J_{\mathrm{export}}, \tag{13}$$

where $\sigma_{\mathrm{int}} \geq 0$ is the internal (irreversible) entropy production rate, and $J_{\mathrm{export}} = (P_\gamma + P_\kappa)/T$ is the entropy current flowing out of the system into the environment.

Proposition 4 (Attractor implies net entropy export)**.** *If a stable attractor exists* ($\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$) *and the attractor is non-trivial* ($\psi_* \neq 0$)*, then*

$$J_{\mathrm{export}} \;>\; \sigma_{\mathrm{int}} \;\geq\; 0. \tag{14}$$

The system is a net entropy exporter.

Argument. At the attractor (fixed point), the system entropy S_{sys} is stationary: $\frac{\mathrm{d}}{\mathrm{d}t}S_{\mathrm{sys}} = 0$. From (13) this gives $\sigma_{\mathrm{int}} = J_{\mathrm{export}}$ exactly at the fixed point — entropy production equals entropy export. Since the attractor is non-trivial ($\psi_* \neq 0$), $P_\gamma = \gamma\|\psi_*\|^2 > 0$, hence $J_{\mathrm{export}} > 0$ and therefore $\sigma_{\mathrm{int}} > 0$.

Throughout the basin of attraction away from the fixed point, the Lyapunov condition $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$ implies that the system is moving toward a lower-entropy ordered state, requiring $J_{\mathrm{export}} > \sigma_{\mathrm{int}}$ to satisfy the second law. The strict inequality $J_{\mathrm{export}} > \sigma_{\mathrm{int}}$ therefore holds throughout the basin except at the fixed point itself, where equality holds. □

Remark 3. *The argument is thermodynamic, not dynamical. It does not depend on the specific form of equations* (1)–(2)*, only on three properties: (i) the system is open, (ii) it has dissipative terms, (iii) it admits a non-trivial stationary state. Any system with these properties must export entropy. The AHT equations make the dissipation channels explicit (γ, κ). Note that this generality applies to Propositions 1–3 only. Theorem 2 is specific to the AHT equations: the exact form of J_{export} in* (16) *depends on the particular dissipation channels γ and κ defined by equations* (1)–(2)*.*

5 Step 4: Necessity—The Closed System Cannot Form Attractors

We now prove the converse: without entropy export, no stable non-trivial attractor can exist.

Theorem 1 (Impossibility of asymptotically stable fixed points in the closed system)**.** *Set $S(t) = 0$ (no input) and $\gamma = 0$ (no damping). Then no fixed point $(\psi_*, \delta L_*)$ with $\psi_* \neq 0$ is asymptotically stable. In particular, no non-trivial attractor of fixed-point type exists.*

Proof. With $S = 0$ and $\gamma = 0$, equation (4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\psi = 0, \tag{15}$$

so $\|\psi(t)\|^2 = \|\psi(0)\|^2$ for all $t \geq 0$: field energy is a conserved quantity.

Suppose for contradiction that $(\psi_*, \delta L_*)$ with $\psi_* \neq 0$ is asymptotically stable. Then there exists an open neighbourhood $U \ni (\psi_*, \delta L_*)$ such that every trajectory starting in U satisfies $(\psi(t), \delta L(t)) \to (\psi_*, \delta L_*)$, in particular $\|\psi(t)\|^2 \to \|\psi_*\|^2$.

However, for any $(\psi_0, \delta L_0) \in U$ with $\|\psi_0\|^2 \neq \|\psi_*\|^2$, conservation (15) gives $\|\psi(t)\|^2 = \|\psi_0\|^2 \neq \|\psi_*\|^2$ for all t, so convergence to ψ_* is impossible. Since states with $\|\psi_0\|^2 \neq \|\psi_*\|^2$ are dense in every neighbourhood of $(\psi_*, \delta L_*)$, no open basin of attraction can exist. This contradicts asymptotic stability. □

Remark 4. *The proof uses only energy conservation for ψ; the behaviour of δL is immaterial. The argument applies even if δL reaches a stationary value (driven by the Hebbian term): even with a fixed δL_*, the field ψ evolves unitarily and cannot converge to a fixed point. The essential asymmetry with the open system is that $\gamma > 0$ breaks energy conservation and makes the ψ dynamics contractive, enabling asymptotic stability.*

6 Step 5: Quantitative Lower Bound

Theorem 2 (Exact entropy export at the attractor)**.** *Let $(\psi_*, \delta L_*)$ be a stable fixed-point attractor of* (1)–(2) *under constant input S_* with $F_* > 0$. Under Assumption 3, the entropy export rate satisfies exactly:*

$$T \cdot J_{\mathrm{export}} \; = \; \gamma\|\psi_*\|^2 + \kappa\|\delta L_*\|_F^2. \tag{16}$$

Expressed purely in terms of ψ_*, using the stationarity relation $\delta L_* = -(\eta F_*/\kappa)\,\mathrm{Re}[\psi_*\psi_*^\dagger]$:

$$T \cdot J_{\text{export}} \;=\; \gamma\|\psi_*\|^2 + \frac{\eta^2 F_*^2}{\kappa}\|\,\mathrm{Re}[\psi_*\psi_*^\dagger]\|_F^2. \tag{17}$$

Proof. At the attractor, $\frac{\mathrm{d}}{\mathrm{d}t}E_\psi = 0$. From (4):

$$\mathrm{Re}\langle S_*, \psi_*\rangle = \gamma\|\psi_*\|^2 \;=:\; P_\gamma. \tag{18}$$

The input power equals the field dissipation; P_γ exits the system as heat (Assumption 3).

At the attractor, $\frac{\mathrm{d}}{\mathrm{d}t}E_L = 0$. From (5):

$$-\eta\, F_*\ \mathrm{tr}\big[\delta L_* \cdot \mathrm{Re}[\psi_*\psi_*^\dagger]\big] = \kappa\|\delta L_*\|_F^2 \;=:\; P_\kappa. \tag{19}$$

The Hebbian drive exactly compensates the κ-decay; P_κ is dissipated (Assumption 3).

The total dissipation is the entropy export rate (times T):

$$T \cdot J_{\text{export}} = P_\gamma + P_\kappa = \gamma\|\psi_*\|^2 + \kappa\|\delta L_*\|_F^2. \tag{20}$$

Equation (17) follows by substituting $\delta L_* = -(\eta F_*/\kappa)\,\mathrm{Re}[\psi_*\psi_*^\dagger]$ (from (8)) into $\kappa\|\delta L_*\|_F^2$. ☐

Remark 5 (Interpretation)**.** *The two terms in* (16) *have distinct metabolic meanings:*

1. *$\gamma\|\psi_*\|^2$: the cost of maintaining the field pattern against damping — the* metabolic cost of representation*.*
2. *$\kappa\|\delta L_*\|_F^2$: the cost of maintaining the Laplacian perturbation against forgetting — the* metabolic cost of working memory*.*

Both costs scale as squared norms of the attractor state. The equality in (16) *is sharp: it is the exact dissipation at the fixed point, not a lower bound. It functions as a lower bound on J_{export} for any trajectory near the attractor because the Lyapunov condition requires additional dissipation away from the fixed point (Section 3).*

7 Step 6: Biological Implication

The human brain has mass $m \approx 1.3\,\mathrm{kg}$ and rests at temperature $T \approx 310\,\mathrm{K}$. It consumes approximately $P_{\text{brain}} \approx 20\,\mathrm{W}$, virtually all of which is dissipated as heat (the brain performs negligible mechanical work). The entropy export rate is therefore:

$$J_{\text{export}}^{\text{brain}} \;=\; \frac{P_{\text{brain}}}{T} \;\approx\; \frac{20}{310} \;\approx\; 0.065\ \mathrm{W/K}. \tag{21}$$

From Theorem 2, this sets an upper bound on the total "attractor load" the brain can sustain simultaneously:

$$\sum_\alpha \left[\gamma\|\psi_\alpha^*\|^2 + \frac{\kappa^2}{\eta F_*}\|\delta L_\alpha^*\|_F^2\right] \;\le\; T \cdot J_{\text{export}}^{\text{brain}} \;\approx\; 20\ \mathrm{W}, \tag{22}$$

where the sum runs over all simultaneously active attractors α.

Interpretive argument. A more conservative estimate partitions P_{brain} into baseline metabolism ($\approx 14\,\mathrm{W}$, (Raichle and Mintun, 2006)) and task-related incremental cost ($\approx$ 1–5% of baseline, (Clarke and Sokoloff, 2014)). The task-related budget $\Delta P \approx 0.7$–$1.0\,\mathrm{W}$ then yields $P_\alpha^{\min} \approx 100$–$140\,\mathrm{mW}$ per attractor — a quantity that is in principle testable with calorimetric fMRI methods. The cruder estimate of $2.9\,\mathrm{W}$ per item (from the full $20\,\mathrm{W}$ budget divided by Miller's $N_{\max} \approx 7$) constitutes an *upper bound*; the true per-attractor cost is likely one to two orders of magnitude smaller.

8 Summary

The argument is:

1. The AHT system is thermodynamically open with identified energy input ($S(t)$) and dissipation (γ, κ) channels.
2. Stable attractors require $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \leq 0$, which implies net dissipation exceeds net input in the basin of attraction.
3. Net dissipation at temperature T is equivalent to net entropy export: $J_{\text{export}} > 0$.
4. Without entropy export (closed system), energy conservation prevents asymptotic stability: no fixed-point attractor with $\psi_* \neq 0$ can attract a neighbourhood.
5. The exact entropy export rate at the attractor is $J_{\text{export}} = T^{-1}[\gamma\|\psi_*\|^2 + \kappa\|\delta L_*\|_F^2]$, which serves as a lower bound throughout the basin of attraction.
6. The 20 W brain metabolism is consistent with this bound and constrains the total attractor capacity.

The answer to the title question is: yes. Stable attractor formation in the AHT equations—and, by extension, in any dissipative open dynamical system far from equilibrium—requires entropy export as a thermodynamic necessity. This is not a contingent feature of the AHT equations but a consequence of the second law of thermodynamics applied to open systems with non-trivial stationary states.

9 Open Questions

1. **Basin-wide dissipation bound.** Equation (16) is the exact dissipation at the fixed point. Can a sharp position-dependent bound be derived throughout the basin of attraction, i.e. $J_{\text{export}}(\psi, \delta L) \geq f(\psi, \delta L, \psi_*, \delta L_*)$ with equality only at the fixed point? Is it tight? A tighter bound would require incorporating the stability margin (how far inside the basin of attraction the system operates) and the spectral gap of the Lyapunov function's Hessian. This may connect to the thermodynamic uncertainty relations of Barato & Seifert (Barato and Seifert, 2015).

2. **Information-theoretic refinement.** The bound is stated in terms of energy dissipation. A stronger statement would bound the mutual information $I(\psi_*; S_*)$—the information the attractor carries about its input—in terms of the entropy export rate. This would connect AHT to the thermodynamics of computation (Landauer, 1961; Still et al., 2012).

3. **Transient attractor formation cost.** The bound (16) describes steady-state maintenance. The *formation* of a new attractor (learning) plausibly requires excess entropy export above the maintenance rate. Quantifying this transient excess would yield a thermodynamic learning cost analogous to the energy cost of bit erasure in Landauer's principle.

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